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On canonical constructions on connections

ABSTRACT. We study how a projectable general connection Γ in a 2-fibred manifold $Y^2 \rightarrow Y^1 \rightarrow Y^0$ and a general vertical connection Θ in $Y^2 \rightarrow Y^1 \rightarrow Y^0$ induce a general connection $A(\Gamma, \Theta)$ in $Y^2 \rightarrow Y^1$.

Introduction. In Section 1, we introduce the concepts of projectable general connections Γ and general vertical connections Θ in a 2-fibred manifold $Y^2 \rightarrow Y^1 \rightarrow Y^0$. In Section 2, we construct a general connection $\Sigma(\Gamma, \Theta)$ in $Y^2 \rightarrow Y^1$ from a projectable general connection Γ in $Y^2 \rightarrow Y^1 \rightarrow Y^0$ by means of a general vertical connection Θ in $Y^2 \rightarrow Y^1 \rightarrow Y^0$. In Section 3 we observe the canonical character of the construction $\Sigma(\Gamma, \Theta)$. In Section 4, we cite the concepts of natural operators. In Section 5, we describe completely the natural operators A transforming tuples (Γ, Θ) as above into general connections $A(\Gamma, \Theta)$ in $Y^2 \rightarrow Y^1$. In Section 6, we prove that there is no natural operator C producing general connections $C(\Gamma)$ in $Y^2 \rightarrow Y^1$ from projectable general connections Γ in $Y^2 \rightarrow Y^1 \rightarrow Y^0$. In Section 7, we present a construction of a general connection $\Sigma(\Gamma, \Theta)$ in $Y^2 \rightarrow Y^1$ from a system $\Gamma = (\Gamma^2, \Gamma^1)$ of a general connection Γ^2 in $Y^2 \rightarrow Y^0$ and a general connection Γ^1 in $Y^1 \rightarrow Y^0$ by means of a general vertical connection Θ in $Y^2 \rightarrow Y^1 \rightarrow Y^0$. In Section 8, we present an application of the obtained result in prolongation of general connections to bundle functors.

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All manifolds considered in the note is Hausdorff, second countable, without boundaries, finite dimensional and smooth (of class C^∞). Maps between manifolds are smooth (infinitely differentiable).

1. Connections. A fibred manifold is a surjective submersion $p : Y \rightarrow M$ between manifolds. By [1], an r -th order holonomic connection in $p : Y \rightarrow M$ is a section

$$\Gamma : Y \rightarrow J^r Y$$

of the holonomic r -jet prolongation $\pi_0^r : J^r Y \rightarrow Y$ of $Y \rightarrow M$. If $Y \rightarrow M$ is a vector bundle and $\Gamma : Y \rightarrow J^r Y$ is a vector bundle map, Γ is called a linear r -th order holonomic connection in $Y \rightarrow M$. A linear r -th order holonomic connection in the tangent bundle $Y = TM \rightarrow M$ of M is called an r -th order linear connection on M . A first order linear connection on M is in fact a classical linear connection on M .

A 1-order holonomic connection $\Gamma : Y \rightarrow J^1 Y$ in a fibred manifold $Y \rightarrow M$ is called a general connection in $Y \rightarrow M$.

We have the following equivalent definitions of general connections in $Y \rightarrow M$, see [1].

A general connection in $p : Y \rightarrow M$ is a lifting map

$$\Gamma : Y \times_M TM \rightarrow TY ,$$

i.e. a vector bundle map covering the identity map $id_Y : Y \rightarrow Y$ such that

$$Tp \circ \Gamma(y, w) = w$$

for any $y \in Y_x$, $w \in T_x M$, $x \in M$. (More precisely, $\Gamma(y, w) = T_x \sigma(w)$, where $\Gamma(y) = j_x^1 \sigma$.)

A general connection in $Y \rightarrow M$ is a vector bundle decomposition

$$TY = VY \oplus_Y H^\Gamma$$

of the tangent bundle TY of Y , where VY is the vertical bundle of Y . (More precisely, $H_y^\Gamma = \text{im } T_x \sigma$, where $\Gamma(y) = j_x^1 \sigma$.)

A general connection in $Y \rightarrow M$ is a vector bundle projection (in direction H^Γ)

$$pr^\Gamma : TY \rightarrow VY$$

covering id_Y .

A 2-fibred manifold is a system $Y^2 \rightarrow Y^1 \rightarrow Y^0$ of two fibred manifolds $Y^2 \rightarrow Y^1$ and $Y^1 \rightarrow Y^0$.

Let $Y^2 \rightarrow Y^1 \rightarrow Y^0$ be 2-fibred manifold and

$$p^{ij} : Y^i \rightarrow Y^j , \quad 0 \leq j < i \leq 2$$

be its projections. Of course, $p^{20} = p^{10} \circ p^{21}$. Let

$$V^{ij} Y^i := \ker(Tp^{ij} : TY^i \rightarrow TY^j)$$

be the vertical bundle of $p^{ij} : Y^i \rightarrow Y^j$, $0 \leq j < i \leq 2$.

We introduce the following concepts of projectable general connections and of general vertical connections in 2-fibred manifolds $Y^2 \rightarrow Y^1 \rightarrow Y^0$.

A projectable general connection in $Y^2 \rightarrow Y^1 \rightarrow Y^0$ is a general connection

$$\Gamma : Y^2 \times_{Y^0} TY^0 \rightarrow TY^2$$

in $p^{20} : Y^2 \rightarrow Y^0$ such that there is a (unique) general connection

$$\underline{\Gamma} : Y^1 \times_{Y^0} TY^0 \rightarrow TY^1$$

in $p^{10} : Y^1 \rightarrow Y^0$ satisfying

$$Tp^{21} \circ \Gamma = \underline{\Gamma} \circ (p^{21} \times id_{TY^0}) .$$

Connection $\underline{\Gamma}$ is called the underlying connection of Γ .

A general vertical connection in $Y^2 \rightarrow Y^1 \rightarrow Y^0$ is a vector bundle map

$$\Theta : Y^2 \times_{Y^1} V^{10}Y^1 \rightarrow V^{20}Y^2$$

covering the identity map $id_{Y^2} : Y^2 \rightarrow Y^2$ such that

$$Tp^{21} \circ \Theta(y^2, v^1) = v^1$$

for any $y^2 \in Y_{y^1}^2$, $y^1 \in Y^1$ and $v^1 \in V_{y^1}^{10}Y^1$.

Equivalently, a general vertical connection in $Y^2 \rightarrow Y^1 \rightarrow Y^0$ is a smoothly parametrized system $\Theta = (\Theta_x)$ of general connections

$$\Theta_x : Y_x^2 \times_{Y_x^1} TY_x^1 \rightarrow TY_x^2$$

in the fibred manifolds $Y_x^2 \rightarrow Y_x^1$ for any $x \in Y^0$, where Y_x^2 is the fibre of $p^{20} : Y^2 \rightarrow Y^0$ over x and Y_x^1 is the fibre of $p^{10} : Y^1 \rightarrow Y^0$ over x and $Y_x^2 \rightarrow Y_x^1$ is the restriction of the projection $p^{21} : Y^2 \rightarrow Y^1$.

2. A construction. Let Γ be a projectable general connection in $Y^2 \rightarrow Y^1 \rightarrow Y^0$ with the underlying connection $\underline{\Gamma}$ and Θ be a general vertical connection in $Y^2 \rightarrow Y^1 \rightarrow Y^0$.

We define a map $\Sigma(\Gamma, \Theta) = \Sigma : Y^2 \times_{Y^1} TY^1 \rightarrow TY^2$ by

$$\Sigma(y^2, w^1) := \Theta(y^2, pr^{\underline{\Gamma}}(w^1)) + \Gamma(y^2, Tp^{10}(w^1)) ,$$

$y^2 \in Y_{y^1}^2$, $y^1 \in Y^1$, $w^1 \in T_{y^1}Y^1$, where $pr^{\underline{\Gamma}} : TY^1 \rightarrow V^{10}Y^1$ is the $\underline{\Gamma}$ -projection.

Lemma 1. Σ is a general connection in $p^{21} : Y^2 \rightarrow Y^1$.

Proof. It is sufficient to verify that $Tp^{21} \circ \Sigma(y^2, w^1) = w^1$. We consider two cases.

(a) Let $w^1 \in V_{y^1}^{10}Y^1$. Then $\Sigma(y^2, w^1) = \Theta(y^2, w^1)$, and then

$$Tp^{21} \circ \Sigma(y^2, w^1) = Tp^{21} \circ \Theta(y^2, w^1) = w^1$$

as Θ is a general vertical connection in $Y^2 \rightarrow Y^1 \rightarrow Y^0$.

(b) Let $w^1 \in H_{y^1}^\Gamma Y^1$, the Γ -horizontal space. Denote $w^0 = Tp^{10}(w^1)$. Then $\Sigma(y^2, w^1) = \Gamma(y^2, w^0)$, and then

$$Tp^{21} \circ \Sigma(y^2, w^1) = Tp^{21} \circ \Gamma(y^2, w^0) = \Gamma(p^{21}(y^2), w^0) = \Gamma(y^1, w^0).$$

Then $w' := Tp^{21} \circ \Sigma(y^2, w^1) \in H_{y^1}^\Gamma Y^1$, $w^1 \in H_{y^1}^\Gamma Y^1$ and

$$Tp^{10}(w') = Tp^{10} \circ Tp^{21} \circ \Gamma(y^2, w^0) = Tp^{20} \circ \Gamma(y^2, w^0) = w^0 = Tp^{10}(w^1),$$

and consequently $w' = w^1$. \square

3. Invariance. Let $\tilde{Y}^2 \rightarrow \tilde{Y}^1 \rightarrow \tilde{Y}^0$ be another 2-fibred manifold with projections $\tilde{p}^{ij} : \tilde{Y}^i \rightarrow \tilde{Y}^j$, $0 \leq j < i \leq 2$. Let $\tilde{\Gamma}$ be a projectable general connection in $\tilde{Y}^2 \rightarrow \tilde{Y}^1 \rightarrow \tilde{Y}^0$ and $\tilde{\Theta}$ be a general vertical connection in $\tilde{Y}^2 \rightarrow \tilde{Y}^1 \rightarrow \tilde{Y}^0$. Let $f = (f^2, f^1, f^0) : (Y^2 \rightarrow Y^1 \rightarrow Y^0) \rightarrow (\tilde{Y}^2 \rightarrow \tilde{Y}^1 \rightarrow \tilde{Y}^0)$ be a 2-fibred map, i.e. $f^i : Y^i \rightarrow \tilde{Y}^i$ for $i = 0, 1, 2$ and $\tilde{p}^{ij} \circ f^i = f^j \circ p^{ij}$ for $0 \leq j < i \leq 2$.

Lemma 2. *If Γ is f -related with $\tilde{\Gamma}$, (i.e. $Tf^2 \circ \Gamma = \tilde{\Gamma} \circ (f^2 \times_{f^0} Tf^0)$) and then $Tf^1 \circ \Gamma = \tilde{\Gamma} \circ (f^1 \times_{f^0} Tf^0)$ and Θ is f -related with $\tilde{\Theta}$ (i.e. $V^{20}f^2 \circ \Theta = \tilde{\Theta} \circ (f^2 \times_{f^1} V^{10}f^1)$), then $\Sigma = \Sigma(\Gamma, \Theta)$ is f -related with $\tilde{\Sigma} = \Sigma(\tilde{\Gamma}, \tilde{\Theta})$ (i.e. $Tf^2 \circ \Sigma = \tilde{\Sigma} \circ (f^2 \times_{f^1} Tf^1)$).*

Proof. If $w \in H^\Gamma Y^1$, then $w = \Gamma(y^1, w^0)$ for some $y^1 \in Y_{y^0}^1$ and $w^0 \in Y_{y^0}^0$, and then $Tf^1(w) = \tilde{\Gamma}(f^1(y^1), Tf^0(w^0)) \in H^{\tilde{\Gamma}}$. Then

$$Tf^1(H^\Gamma Y^1) \subset H^{\tilde{\Gamma}} \tilde{Y}^1 \text{ and (obviously) } Tf^1(V^{10}Y^1) \subset V^{10}\tilde{Y}^1.$$

Consequently, $V^{10}f^1 \circ pr^\Gamma = pr^{\tilde{\Gamma}} \circ Tf^1$. Using this formula and the assumption of the lemma and the formula defining Σ , one can easily verify that

$$Tf^2 \circ \Sigma(y^2, w^1) = \tilde{\Sigma} \circ (f^2(y^2), Tf^1(w^1))$$

for $y^2 \in Y_{y^1}^2$, $w^1 \in T_{y^1}Y^1$, $y^1 \in Y^1$. \square

4. Natural operators. The general concept of natural operators can be found in [1]. We need the following partial cases of this general concept.

Let $\mathcal{FM}_{m_0, m_1, m_2}$ be the category of 2-fibred manifolds $Y^2 \rightarrow Y^1 \rightarrow Y^0$ with $\dim(Y^0) = m_0$, $\dim(Y^1) = m_0 + m_1$, $\dim(Y^2) = m_0 + m_1 + m_2$ and their 2-fibred local diffeomorphisms.

Definition 1. An $\mathcal{FM}_{m_0, m_1, m_2}$ -natural operator transforming projectable general connections Γ and general vertical connections Θ in $\mathcal{FM}_{m_0, m_1, m_2}$ -objects $Y^2 \rightarrow Y^1 \rightarrow Y^0$ into general connections $A(\Gamma, \Theta)$ in $Y^2 \rightarrow Y^1$ is an $\mathcal{FM}_{m_0, m_1, m_2}$ -invariant system A of regular operators (functions)

$$A : Con_{proj}(Y^2 \rightarrow Y^1 \rightarrow Y^0) \times Con_{vert}(Y^2 \rightarrow Y^1 \rightarrow Y^0) \rightarrow Con(Y^2 \rightarrow Y^1)$$

for any $\mathcal{FM}_{m_0, m_1, m_2}$ -objects $Y^2 \rightarrow Y^1 \rightarrow Y^0$, where $Con_{proj}(Y^2 \rightarrow Y^1 \rightarrow Y^0)$ is the set of projectable general connections in $Y^2 \rightarrow Y^1 \rightarrow Y^0$, $Con_{vert}(Y^2 \rightarrow Y^1 \rightarrow Y^0)$ is the set of general vertical connections in $Y^2 \rightarrow Y^1 \rightarrow Y^0$ and $Con(Y^2 \rightarrow Y^1)$ is the set of general connections in $Y^2 \rightarrow Y^1$.

The invariance of A means that if $\Gamma \in Con_{proj}(Y^2 \rightarrow Y^1 \rightarrow Y^0)$ is f -related with $\tilde{\Gamma} \in Con_{proj}(\tilde{Y}^2 \rightarrow \tilde{Y}^1 \rightarrow \tilde{Y}^0)$ and $\Theta \in Con_{vert}(Y^2 \rightarrow Y^1 \rightarrow Y^0)$ is f -related with $\tilde{\Theta} \in Con_{vert}(\tilde{Y}^2 \rightarrow \tilde{Y}^1 \rightarrow \tilde{Y}^0)$ for an $\mathcal{FM}_{m_0, m_1, m_2}$ -morphism $f = (f^2, f^1, f^0) : (Y^2 \rightarrow Y^1 \rightarrow Y^0) \rightarrow (\tilde{Y}^2 \rightarrow \tilde{Y}^1 \rightarrow \tilde{Y}^0)$, then $A(\Gamma, \Theta)$ is f -related with $A(\tilde{\Gamma}, \tilde{\Theta})$.

The regularity of A means that A transforms smoothly parametrized families into smoothly parametrized families.

Because of Lemma 2, the construction $\Sigma(\Gamma, \Theta)$ defines an $\mathcal{FM}_{m_0, m_1, m_2}$ -natural operator in the sense of Definition 1. So, to describe all natural operators A in the sense of Definition 1 it is sufficient to describe all natural operators in the sense of the following definition.

Definition 2. An $\mathcal{FM}_{m_0, m_1, m_2}$ -natural operator transforming projectable general connections Γ and general vertical connections Θ in $\mathcal{FM}_{m_0, m_1, m_2}$ -objects $Y^2 \rightarrow Y^1 \rightarrow Y^0$ into sections $B(\Gamma, \Theta) : Y^2 \rightarrow T^*Y^1 \otimes V^{21}Y^2$ of $T^*Y^1 \otimes V^{21}Y^2 \rightarrow Y^2$ is an $\mathcal{FM}_{m_0, m_1, m_2}$ -invariant system A of regular operators

$$B : Con_{proj}(Y^2 \rightarrow Y^1 \rightarrow Y^0) \times Con_{vert}(Y^2 \rightarrow Y^1 \rightarrow Y^0) \rightarrow C_{Y^2}^\infty(T^*Y^1 \otimes V^{21}Y^2)$$

for any $\mathcal{FM}_{m_0, m_1, m_2}$ -object $Y^2 \rightarrow Y^1 \rightarrow Y^0$, where $C_{Y^2}^\infty(T^*Y^1 \otimes V^{21}Y^2)$ is the space of sections of the vector bundle $T^*Y^1 \otimes V^{21}Y^2$ over Y^2 (with respect to the clear projection).

It is obvious that any natural operator A in the sense of Definition 1 is of the form

$$A(\Gamma, \Theta) = \Sigma(\Gamma, \Theta) + B(\Gamma, \Theta)$$

for a uniquely determined (by A) natural operator B in the sense of Definition 2.

A simple example of a natural operator in the sense of Definition 2 is the one B^o defined by

$$B^o(\Gamma, \Theta)(y^2)(w^1) = pr^{\Sigma(\Gamma, \Theta)} \circ \Theta(y^2, pr^\Gamma(w^1)) \in V_{y^2}^{21}Y^2$$

for any $\mathcal{FM}_{m_0, m_1, m_2}$ -object $Y^2 \rightarrow Y^1 \rightarrow Y^0$, $\Gamma \in Con_{proj}(Y^2 \rightarrow Y^1 \rightarrow Y^0)$, $\Theta \in Con_{vert}(Y^2 \rightarrow Y^1 \rightarrow Y^0)$, $y^2 \in Y_{y^1}^2$, $y^1 \in Y^1$, $w^1 \in T_{y^1}Y^1$, where $pr^{\Sigma(\Gamma, \Theta)} : TY^2 \rightarrow V^{21}Y^2$ is the $\Sigma(\Gamma, \Theta)$ -projection.

5. A classification. Let $\mathbf{R}^{m_0, m_1, m_2}$ be the trivial $\mathcal{FM}_{m_0, m_1, m_2}$ -object $\mathbf{R}^{m_0} \times \mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \rightarrow \mathbf{R}^{m_0} \times \mathbf{R}^{m_1} \rightarrow \mathbf{R}^{m_0}$ with the usual projections. Let $x^1, \dots, x^{m_0}, y^1, \dots, y^{m_1}, z^1, \dots, z^{m_2}$ be the usual coordinates on $\mathbf{R}^{m_0, m_1, m_2}$.

Consider a natural operator B in the sense of Definition 2. Because of the invariance of B with respect to 2-fibred manifold charts, B is determined by the linear maps

$$B(\Gamma, \Theta)(0, 0, 0) : T_{(0,0)}(\mathbf{R}^{m_0} \times \mathbf{R}^{m_1}) \rightarrow V_{(0,0,0)}^{21}(\mathbf{R}^{m_0} \times \mathbf{R}^{m_1} \times \mathbf{R}^{m_2})$$

for all $\Gamma \in \text{Con}_{\text{proj}}(\mathbf{R}^{m_0, m_1, m_2})$ and all $\Theta \in \text{Con}_{\text{vert}}(\mathbf{R}^{m_0, m_1, m_2})$ of the forms

$$\Gamma = \Gamma^o + \sum \Gamma_i^p(x, y) dx^i \otimes \frac{\partial}{\partial y^p} + \sum \Gamma_i^q(x, y, z) dx^i \otimes \frac{\partial}{\partial z^q},$$

$$\Theta = \Theta^o + \sum \Theta_p^q(x, y, z) dy^p \otimes \frac{\partial}{\partial z^q},$$

where the sums are over $i = 1, \dots, m_0$, $p = 1, \dots, m_1$, $q = 1, \dots, m_2$, and where Γ^o denotes the trivial projectable general connection in $\mathbf{R}^{m_0, m_1, m_2}$ and $\Theta^o = \sum dy^p \otimes \frac{\partial}{\partial y^p}$ denotes the trivial general vertical connection in $\mathbf{R}^{m_0, m_1, m_2}$.

Eventually, using a new 2-fibred manifold chart one can additionally assume $\Gamma_i^p(0, 0) = 0$ and $\Gamma_i^q(0, 0, 0) = 0$. (More precisely, denote $j_0^1 \sigma := \Gamma(0, 0, 0)$ and $\sigma(x) =: (x, \tilde{\sigma}(x), \bar{\sigma}(x))$. We consider the 2-fibred coordinate system $(x, y - \tilde{\sigma}(x), z - \bar{\sigma}(x))$. In the coordinate system $\Gamma(0, 0, 0) = \Gamma^o(0, 0, 0)$.)

Then using the invariance of B with respect to $\mathcal{FM}_{m_0, m_1, m_2}$ -map $\frac{1}{t} id$ for $t > 0$ and then putting $t \rightarrow 0$, we can assume $\Gamma = \Gamma^o$ and $\Theta_p^q(x, y, z) = \Theta_p^q(0, 0, 0) = \text{const.}$ Consequently, B is determined by the maps

$$B\left(\Gamma^o, \Theta^o + \sum \Theta_p^q dy^p \otimes \frac{\partial}{\partial z^q}\right)(0, 0, 0) : \mathbf{R}^{m_0} \times \mathbf{R}^{m_1} \rightarrow \mathbf{R}^{m_2}$$

for all $\Theta_p^q \in \mathbf{R}$, $p = 1, \dots, m_1$, $q = 1, \dots, m_2$.

Using the invariance of B with respect to $t id_{\mathbf{R}^{m_0}} \times id_{\mathbf{R}^{m_1}} \times id_{\mathbf{R}^{m_2}}$ and then putting $t \rightarrow 0$, we deduce that $B(\Gamma^o, \Theta^o + \sum \Theta_p^q dy^p \otimes \frac{\partial}{\partial z^q})(0, 0, 0)$ do not depend on elements from \mathbf{R}^{m_0} . Consequently, B is determined by the map $\Phi : \mathbf{R}^{m_1*} \otimes \mathbf{R}^{m_2} \rightarrow \mathbf{R}^{m_1*} \otimes \mathbf{R}^{m_2}$ given by

$$\Phi((\Theta_p^q)) = B\left(\Gamma^o, \Theta^o + \sum \Theta_p^q dy^p \otimes \frac{\partial}{\partial z^q}\right)(0, 0, 0) \in \mathbf{R}^{m_1*} \otimes \mathbf{R}^{m_2}.$$

Using the invariance of B with respect to linear isomorphisms from $\{id_{\mathbf{R}^{m_0}}\} \times GL(m_1) \times GL(m_2)$, we deduce that Φ is $GL(m_1) \times GL(m_2)$ -invariant. Consequently, Φ is the constant multiple of the identity. Then the space of all $\mathcal{FM}_{m_0, m_1, m_2}$ -natural operators B in the sense of Definition 2 is 1-dimensional. So, any natural operator B in the sense of Definition 2 is the constant multiple of B^o .

Thus we proved the following classification theorem.

Theorem 1. Any $\mathcal{FM}_{m_0, m_1, m_2}$ -natural operator A in the sense of Definition 1 is of the form

$$A(\Gamma, \Theta) = \Sigma(\Gamma, \Theta) + \tau B^o(\Gamma, \Theta)$$

for a uniquely (by A) real number τ .

6. Why do we use auxiliary a general vertical connection? We prove the following theorem.

Theorem 2. There is no $\mathcal{FM}_{m_0, m_1, m_2}$ -natural operator

$$C : \text{Con}_{\text{proj}}(Y^2 \rightarrow Y^1 \rightarrow Y^0) \rightarrow \text{Con}(Y^2 \rightarrow Y^1)$$

transforming projectable general connections Γ in $\mathcal{FM}_{m_0, m_1, m_2}$ -objects $Y^2 \rightarrow Y^1 \rightarrow Y^0$ into general connections $C(\Gamma)$ in $Y^2 \rightarrow Y^1$.

Proof. Suppose that such C exists. Let Γ^o be the trivial projectable general connection in the 2-fibred manifold $\mathbf{R}^{m_0, m_1, m_2}$. Then $C(\Gamma^o)$ is φ -invariant by any $\mathcal{FM}_{m_0, m_1, m_2}$ -map φ of the form $\varphi(x_0, x_1, x_2) = (x_0, \varphi_1(x_1), \varphi_2(x_1, x_2))$, $x_0 \in \mathbf{R}^{m_0}$, $x_1 \in \mathbf{R}^{m_1}$, $x_2 \in \mathbf{R}^{m_2}$ (as Γ^o is). Then $j_{(0,0)}^1 \sigma := C(\Gamma^o)(0, 0, 0)$ is φ -invariant for any φ as above with $\varphi(0, 0, 0) = (0, 0, 0)$. Then for $\varphi_1(x_1) = x_1$ and $\varphi_2(x_1, x_2) = x_2 + (x_1^1, 0, \dots, 0)$ we get $j_{(0,0)}^1(\varphi \circ \sigma) = j_{(0,0)}^1 \sigma$, i.e. $j_{(0,0)}^1 \eta = 0$, where $\eta(x_0, x_1) = (x_0, x_1, x_1^1, 0, \dots, 0)$. Contradiction. \square

So, to construct canonically a general connection in $Y^2 \rightarrow Y^1$ from a projectable general connection in $Y^2 \rightarrow Y^1 \rightarrow Y^0$ the using of auxiliary objects is unavoidable. In the present note we have used general vertical connections as such auxiliary ones.

7. A generalization. Let $Y^2 \rightarrow Y^1 \rightarrow Y^0$ be a 2-fibred manifold.

A projectable general connection Γ in $Y^2 \rightarrow Y^1 \rightarrow Y^0$ is in fact a system $\Gamma = (\Gamma, \underline{\Gamma})$ of two general connections in $p^{20} : Y^2 \rightarrow Y^0$ and $p^{10} : Y^1 \rightarrow Y^0$ (respectively), and $\underline{\Gamma}$ is determined by Γ .

In this section, we present how to extend the construction of $\Sigma(\Gamma, \Theta)$ for $\Gamma = (\Gamma, \underline{\Gamma})$ into a construction $\Sigma(\Gamma, \Theta)$ for $\Gamma = (\Gamma^2, \Gamma^1)$, where $\Gamma^2 : Y^2 \times_{Y^0} TY^0 \rightarrow TY^2$ is a general connection in $p^{20} : Y^2 \rightarrow Y^0$ and $\Gamma^1 : Y^1 \times_{Y^0} TY^0 \rightarrow TY^1$ is a general connection in $p^{10} : Y^1 \rightarrow Y^0$.

Let $\Gamma = (\Gamma^2, \Gamma^1)$ and Θ be in question. We define a map $\Sigma(\Gamma, \Theta) = \Sigma : Y^2 \times_{Y^1} TY^1 \rightarrow TY^2$ by

$$\Sigma(y^2, w^1) := \Theta(y^2, pr^{\Gamma^1}(w^1)) + \Gamma^2(y^2, w^0) - \Theta(y^2, pr^{\Gamma^1} \circ Tp^{21} \circ \Gamma^2(y^2, w^0)) ,$$

$$y^2 \in Y_{y^1}^2, y^1 \in Y^1, w^1 \in T_{y^1} Y^1, w^0 = Tp^{10}(w^1) .$$

Lemma 3. Σ is a general connection in $p^{21} : Y^2 \rightarrow Y^1$.

Proof. We are going to prove that $Tp^{21} \circ \Sigma(y^2, w^1) = w^1$. We consider two cases.

(a) Let $w^1 \in V_{y^1}^{10}Y^1$. Then $\Sigma(y^2, w^1) = \Theta(y^2, w^1)$, and next we proceed as in the part (a) of the proof of Lemma 1.

(b) Let $w^1 \in H_{y^1}^{\Gamma^1}Y^1$. Then

$$\Sigma(y^2, w^1) = \Gamma^2(y^2, w^0) - \Theta(y^2, pr^{\Gamma^1} \circ Tp^{21} \circ \Gamma^2(y^2, w^0)) ,$$

and then

$$Tp^{21} \circ \Sigma(y^2, w^1) = Tp^{21} \circ \Gamma^2(y^2, w^0) - pr^{\Gamma^1} \circ Tp^{21} \circ \Gamma^2(y^2, w^0) .$$

So, $w' := Tp^{21} \circ \Sigma(y^2, w^1) \in H_{y^1}^{\Gamma^1}Y^1$ and $w^1 \in H_{y^1}^{\Gamma^1}Y^1 \in H_{y^1}^{\Gamma^1}Y^1$ and

$$Tp^{10}(w') = Tp^{20} \circ \Gamma^2(y^2, w^0) - 0 = w^0 = Tp^{10}(w^1) ,$$

and consequently $w' = w^1$. \square

8. An application. We can use the construction $\Sigma(\Gamma, \Theta)$ from the previous section in prolongation of connections to bundle functors.

Namely, let $F : \mathcal{FM}_{m,n} \rightarrow \mathcal{FM}$ be a bundle functor in the sense of [1] of order r , where \mathcal{FM} is the category of fibred manifolds and fibred maps and $\mathcal{FM}_{m,n}$ is the category of fibred manifolds with m -dimensional bases and n -dimensional fibres and their local fibred diffeomorphisms. Let $p : Y \rightarrow M$ be an $\mathcal{FM}_{m,n}$ -object. Let Ξ be a general connection in $p : Y \rightarrow M$ and λ be an r -th order linear connection on M (i.e. r -th order linear connection in $TM \rightarrow M$). Thus we have the F -prolongation $\mathcal{F}(\Xi, \lambda)$ (of Ξ with respect to λ) in the sense of [1, Def. 45.4]. $\mathcal{F}(\Xi, \lambda)$ is a general connection in $FY \rightarrow M$. Let λ^1 be an r -th order linear connection in $VY \rightarrow Y$. Using the construction $\Sigma(\Gamma, \Theta)$ from the previous section, we can construct a general connection $\mathcal{F}(\Xi, \lambda^1, \lambda)$ in $FY \rightarrow Y$ as follows.

Let $Y^2 = FY \rightarrow Y^1 = Y \rightarrow Y^0 = M$ be the 2-fibred manifold. We have a general vertical connection $\Theta = \Theta(\lambda^1) : Y^2 \times_{Y^1} V^{10}Y^1 \rightarrow V^{20}Y^2$ in $Y^2 \rightarrow Y^1 \rightarrow Y^0$ by

$$\Theta(\lambda^1)(y^2, v^1) := \mathcal{F}X(y^2) , \quad j_{y^1}^r(X) := \lambda^1(v^1) ,$$

$y^2 \in Y_{y^1}^2$, $y^1 \in Y^1$, $v^1 \in V_{y^1}^{10}Y^1$, where $\mathcal{F}X$ is the flow lift of X with respect to F . Denote $\Gamma = (\mathcal{F}(\Xi, \lambda), \Xi)$. Consequently, we have a general connection $\mathcal{F}(\Xi, \lambda, \lambda^1)$ in $FY \rightarrow Y$ by

$$\mathcal{F}(\Xi, \lambda, \lambda^1) := \Sigma(\Gamma, \Theta(\lambda^1)) .$$

Let Ξ and λ be as above and Λ be an r -th order linear connection on Y (i.e. r -th order linear connection in $TY \rightarrow Y$). Using the above construction $\mathcal{F}(\Xi, \lambda, \lambda^1)$, we can construct a general connection $\mathcal{F}(\Xi, \lambda, \Lambda)$ in $FY \rightarrow Y$ as follows.

We have an r -th order linear connection $\lambda^1 = \lambda^1(\Lambda, \Xi)$ in $VY \rightarrow Y$ by

$$\lambda^1(v) = j_y^r(pr^{\Xi} \circ X) , \quad j_y^r X := \Lambda(v) , \quad v \in V_y Y , \quad y \in Y ,$$

where $pr^\Xi : TY \rightarrow VY$ is the Ξ -projection. Then we have a general connection $\mathcal{F}(\Xi, \lambda, \Lambda)$ in $FY \rightarrow Y$ by

$$\mathcal{F}(\Xi, \lambda, \Lambda) := \mathcal{F}(\Xi, \lambda, \lambda^1(\Lambda, \Xi)) .$$

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